Gauge transformations of second type and their implementation. I. Fermions

J. F. Gille* and J. Manuceau†

Centre de Physique Théorique, C.N.R.S., 13-Marseille 9e, France

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A necessary and sufficient condition for implementation of some local gauge transformations in a class of irreducible representations of the CAR algebra is proved. Some particular results on the unitary group of implementation are then given. Not all of the pure states induced by these representations are unitarily equivalent to "quasifree" states of the class we consider; it is shown that such a state is unitarily equivalent to a quasifree state if and only if a certain property (characterizing the "discrete" states) holds.

I. PRELIMINARIES

A. The fermion C^*-algebra and some of its gauge transformations of second type

Let (H, s) be a real separable Hilbert space. Consider the CAR algebra $\mathfrak{A} \equiv \overline{\mathfrak{A}(H,s)}$ built on (H, s), i.e., the C^*-algebra generated by the elements $B(\psi)$, where B is a one-to-one linear map of H into $\mathfrak{A}$ such that

$$[B(\psi), B(\varphi)]_+ = 2s(\psi, \varphi)I \quad \forall \psi, \varphi \in H$$

(I the identity element on $\mathfrak{A}$).

Suppose Λ is a linear operator on H such that

(i) $\dim(\ker \Lambda)$ is not odd (this is not a restriction).

(ii) $|\Lambda|$ is diagonalizable (where $\Lambda = J_0|\Lambda|$ in the polar decomposition).

We choose a complex structure J of H such that

$$\begin{cases} J|(\ker \Lambda)^\perp = J_0|(\ker \Lambda)^\perp, \\ J|\ker \Lambda \text{ an arbitrary complex structure of } \ker \Lambda. \end{cases}$$

We shall write

$$|\Lambda| = \sum_{k\in\mathbb{N}} \lambda_k P_{H_k} \lambda_k \in \mathbb{R}$$

where P_{H_k} are the orthogonal projections on H_k and H_k a two-dimensional real subspace of H which is invariant by J and such that $H = \bigoplus_{k\in\mathbb{N}} H_k$. We remark that some λ_k are possibly not different. (From now we denote by $\bigoplus$ the Hilbert sum and by $\oplus$ the weak sum). Λ is the infinitesimal generator of a one-parameter strongly continuous orthogonal group $\{T_\theta\}_{\theta\in\mathbb{R}}$ on H. By Ref. 1 we can define an automorphism τ_θ of $\mathfrak{A}$ with

$$\tau_\theta(B(\psi)) = B(T_\theta \psi).$$

B. The problem

We look for irreducible representations of $\mathfrak{A}$ for which τ_θ is implementable.

This problem was approached by Dell'Antonio.[2] We give here full proofs of the results announced by him and we generalize some of them.

II. THE CLASS OF REPRESENTATIONS WE CONSIDER

Let $\{\psi_k^1, \psi_k^2\}$ be an orthonormal basis of H_k; then $\Theta_k = -iB(\psi_k^1)B(\psi_k^2)$ verifies

$$[\Theta_k, B(\varphi)]_+ = 0 \quad \forall \varphi \in H_k, \qquad \Theta_k^2 = 1. \tag{II. 1}$$

The center of $\mathfrak{A}_k \equiv \mathfrak{A}(H_k, s)$ is reduced to the scalars, and therefore any solution of (II. 1) is Θ_k or $-\Theta_k$.

Let π'_k be an arbitrary irreducible representation of $\mathfrak{A}_k$ into $\mathcal{H}_k = C^2$.

We construct the representation π' of $\mathfrak{A}$ into $\mathcal{H} = \bigotimes_{k\in\mathbb{N}} \mathcal{H}_k$ from the following: $\forall k \in \mathbb{N}, j = 1, 2,$

$$\pi'(B(\psi_k^j)) = \bigotimes_{j\,1}^{k-1} \pi'_j(\epsilon_j \Theta_j) \otimes \pi'_k(B(\psi_k^j)) \otimes \bigotimes_{j\,k+1}^{\infty} \times I_j (I_j = I_{C^2}),\ \epsilon_j = \pm 1.$$

It is well known that each $\Omega = \bigotimes_{k\in\mathbb{N}} \Omega_k$, Ω_k being a unitary vector of $\mathcal{H}_k$, determines an incomplete tensor product $\mathcal{H}^\Omega = \bigotimes_{k\in\mathbb{N}}^{\mathcal{C}(\Omega)} \mathcal{H}_k$ with $\mathcal{C}(\Omega)$ the equivalence class of Ω for the relation $\Omega \approx \Omega'$ iff

$$\sum_{k\in\mathbb{N}} |(\Omega_k|\Omega'_k) - 1| < +\infty.$$

It is not difficult to see that the $\mathcal{H}^\Omega$ are invariant subspaces of π' and that the restrictions of π' to those subspaces denoted by π'_Ω are irreducible and therefore π' is the direct sum of the set of the π'_Ω.

Let π be the representation of $\mathfrak{A}$ into $\mathcal{H}$ defined by

$$\pi(B(\psi_k^j)) \bigotimes_{l\,1}^{k-1} \pi_l(\Theta_l) \otimes \pi_k(B(\psi_k^j)) \otimes \bigotimes_{l\,k+1}^{\infty} I_l, \quad j = 1, 2,$$

where

$$\pi_l(\Theta_l) = \sigma_l^3, \qquad \pi_k(B(\psi_k^j)) = \sigma_k^j$$

and

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the matrix of σ_l^3 in the canonical basis of $\mathcal{H}_l$,

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the matrix of σ_l^1 in the canonical basis of $\mathcal{H}_l$,

$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is the matrix of σ_l^2 in the canonical basis of $\mathcal{H}_l$.

Accordingly we shall write $\pi = \bigotimes_{k\in\mathbb{N}}^{\circ} \pi_k$.[3] It is clear that for each $l \in \mathbb{N}$, a unitary operator U_l on $\mathcal{H}_l$ exists such that $\forall x \in \mathfrak{A}_l$, $\pi_l(x) = U_l \pi'_l(x) U_l^*$. If $U = \bigotimes_{l\in\mathbb{N}} U_l$, we construct

$$\pi''(x) = U\pi'(x)U^*, \quad \forall x \in \mathfrak{A};$$

hence

$$\pi''(B(\psi_k)) = \bigotimes_{l\,1}^{k-1} \epsilon_l \sigma_l^3 \otimes \pi_k(B(\psi_k)) \otimes \bigotimes_{l\,k+1}^{\infty} I_l.$$

$V = \bigotimes_{l\in\mathbb{N}} V_l$ with $V_l = \sigma_l^3$ if the number of $k < l$ such that $\epsilon_k = -1$ is odd and $V_l = I_l$ otherwise.

Clearly $\pi(x) = V\pi''(x)V^*$, $\forall x \in \mathfrak{A}$; hence $\pi'(x) = W\pi(x)W^*$, $\forall x \in \mathfrak{A}$, where W is a unitary operator on $\mathcal{H}$.

Any irreducible subrepresentation π'_{Ω} of π' is unitary equivalent to the subrepresentation $\pi_{W^*\Omega}$ of π. Therefore we can restrict our attention to the study of the irreducible subrepresentations of π.

Proposition: π_Ω is unitarily equivalent to $\pi_{\Omega'}$ if and only if Ω and Ω' are weakly equivalent.

Proof: Recall that $\Omega = \bigotimes_{k\in\mathbb{N}} \Omega_k$ and $\Omega' = \bigotimes_{k\in\mathbb{N}} \Omega'_k$ are weakly equivalent iff $\sum_{k\in\mathbb{N}} |(|(\Omega_k|\Omega'_k)| - 1)| < +\infty$.

Suppose that Ω and Ω' are weakly equivalent. By Ref. 4, one can find for each $k \in \mathbb{N}$, $\nu_k \in \mathbb{R}$ such that

$$(\Omega'_k)_{k\in\mathbb{N}} \approx (e^{i\nu_k}\Omega_k)_{k\in\mathbb{N}}.$$

Let $U = \bigotimes_{k\in\mathbb{N}} e^{i\nu_k} I_k$. Then $U\Omega \in \mathcal{H}^{\Omega'}$ and we have

$$\pi_{\Omega'}(x) = U\pi_\Omega(x)U^*, \quad \forall x \in \mathfrak{A}.$$

Conversely, if Ω and Ω' are not weakly equivalent, let us denote

$$\omega_\Omega(x) = (\Omega|\pi_\Omega(x)\Omega), \quad x \in \mathfrak{A},$$

and

$$\omega_{\Omega'}(x) = (\Omega'|\pi_{\Omega'}(x)\Omega').$$

Let $U_k \in \mathcal{L}(\mathcal{H}_k)$ be an unitary operator such that

$$U_k\Omega'_k = \Omega_k,$$

and let

$$U'_k = \bigotimes_{j\,1}^{k-1} I_j \otimes U_k \otimes \bigotimes_{j\,k+1}^{\infty} I_j,$$

$$u_k = \pi^{-1}(U'_k).$$

The proof will be continued in the same way as in Sec. IIIA2.

III. THE THEOREM

We note

$$\Omega = \bigotimes_{k\in\mathbb{N}} \begin{pmatrix}\alpha_k\\ \beta_k\end{pmatrix} \quad \text{and} \quad x_k = |\alpha_k|^2.$$

A. Statement

A one-particle evolution τ_θ is implementable for the representation π_Ω if and only if the following condition holds:

$$\text{(A)} \quad \sum_{k\in\mathbb{N}} x_k(1-x_k)\inf(1,\lambda_k^2) < +\infty.$$

If this occurs, a strongly continuous one-parameter group of unitary operators (we shall call such groups SCOPUG) $\{W_\theta\}_{\theta\in\mathbb{R}}$, $W_\theta \in \pi_\Omega(\mathfrak{A})'' = \mathcal{L}(\mathcal{H}^\Omega)$, exists such that

$$\forall x \in \mathfrak{A},\ \forall\theta\in\mathbb{R} \quad \pi_\Omega(\tau_\theta(x)) = W_\theta\pi_\Omega(x)W_{-\theta}.$$

B. Proof

1. Sufficiency

Suppose

$$\sum_{k\in\mathbb{N}} x_k(1-x_k)\inf(1,\lambda_k^2) < +\infty.$$

Let

$$U_{k,\theta} = \begin{pmatrix}1 & 0\\ 0 & e^{i\lambda_k\theta}\end{pmatrix}.$$

It is a unitary operator on $\mathcal{H}_k$. $U_\theta = \bigotimes_{k\in\mathbb{N}} U_{k,\theta}$ is a unitary operator on $\mathcal{H}$.[5] Clearly

$$\pi(\tau_\theta(B(\psi_k^i))) = U_\theta\pi(B(\psi_k^i))U_\theta^{-1}, \quad i = 1,2,\ k\in\mathbb{N};$$

hence U_θ implements τ_θ for the representation π. Changing $U_{k,\theta}$ into $V_{k,\theta} = e^{i\mu_k}U_{k,\theta}$, $\mu_k\in\mathbb{R}$, $V_\theta = \bigotimes_{k\in\mathbb{N}} V_{k,\theta}$ implements τ_θ.

We choose μ_k such that

$$\arg(V_{k,\theta}\Omega_k|\Omega_k) = 0.$$

We get

$$(V_{k,\theta}\Omega_k|\Omega_k)^2 = |(U_{k,\theta}\Omega_k|\Omega_k)|^2 = 1 - 4x_k(1-x_k)\sin^2(\lambda_k\theta/2);$$

from the hypothesis

$$\sum_{k\in\mathbb{N}} x_k(1-x_k)\sin^2(\lambda_k\theta/2) < +\infty;$$

hence

$$\sum_{k\in\mathbb{N}} |(V_{k,\theta}\Omega_k|\Omega_k)^2 - 1| < +\infty.$$

$\prod_{k\in\mathbb{N}}(V_{k,\theta}\Omega_k|\Omega_k)^2$ converges and $V_\theta\mathcal{H}^\Omega \subset \mathcal{H}^\Omega$. We note now V_θ its restriction to $\mathcal{H}^\Omega$. Hence

$$\pi_\Omega(\tau_\theta(x)) = V_\theta\pi_\Omega(x)V_\theta^*, \quad \forall x\in\mathfrak{A}, \quad \text{holds.}$$

It is important to remark that V_θ has been calculated for each $\theta\in\mathbb{R}$ so that $\{V_\theta\}_{\theta\in\mathbb{R}}$ is *not* a group in the general case. By Ref. 6 there exists a SCOPUG $\{W_\theta\}_{\theta\in\mathbb{R}}$ in $\mathcal{L}(\mathcal{H}^\Omega)$ such that

$$\forall x\in\mathfrak{A},\ \forall\theta\in\mathbb{R}, \quad \pi_\Omega(\tau_\theta(x)) = W_\theta\pi_\Omega(x)W_{-\theta}.$$

2. Necessity

Condition (A) is equivalent to the both following conditions:

$$\text{(i)} \quad \sum_{k,|\lambda_k|\geq 1} x_k(1-x_k) < +\infty,$$

$$\text{(ii)} \quad \sum_{k,|\lambda_k|\leq 1} x_k(1-x_k) < +\infty.$$

Suppose condition (A) is false. Then either (i) or (ii) is false. The following two lemmas prove that in the both cases $\exists\theta\in\mathbb{R}$ and $\sum_{k\in\mathbb{N}} x_k(1-x_k)\sin^2(\lambda_k\theta/2) = +\infty$.

Lemma 2.1: Let $(r_k)_{k\in\mathbb{N}}$, $0\leq r_k\leq 1$, and let $(\lambda_k)_{k\in\mathbb{N}}$, $\lambda_k\in\mathbb{R}$, $|\lambda_k|\geq 1$. Then

$$\left(\sum_{k\in\mathbb{N}} r_k\sin^2(\lambda_k\theta) < +\infty,\ \forall\theta\in\mathbb{R}\right) \Rightarrow \sum_{k\in\mathbb{N}} r_k < +\infty.$$

Proof: In our case we have, for $r_k = 4x_k(1-x_k)$,

$$\sum_{k\in\mathbb{N}} r_k\sin^2(\lambda_k\theta) < +\infty, \quad \forall\theta\in\mathbb{R}.$$

In the proof of the sufficient condition we saw that the convergence of this series implies the existence of an SCOPUG $\{W_\theta\}_{\theta\in\mathbb{R}}$, $W_\theta\in\mathcal{L}(\mathcal{H}^\Omega)$ such that

$$\forall x \in \mathfrak{A},\ \forall \theta \in \mathbb{R}, \quad \pi_\Omega(\tau_\theta(x)) = W_\theta \pi_\Omega(x) W_\theta^*.$$

Now we constructed a set of unitary operators $\{V_\theta\}_{\theta\in\mathbb{R}}$ such that

$$\forall x \in \mathfrak{A},\ \forall \theta \in \mathbb{R}, \quad \pi_\Omega(\tau_\theta(x)) = V_\theta \pi_\Omega(x) V_\theta^*.$$

π_Ω being an irreducible representation,

$$W_\theta = \chi(\theta) V_\theta; \quad \chi : \mathbb{R} \to \mathbb{C},\ |\chi(\theta)| = 1;$$

hence

$$|(W_\theta \Omega|\Omega)| = |\chi(\theta)|\,|(V_\theta \Omega|\Omega)| = |(V_\theta\Omega|\Omega)|$$

and

$$|(W_\theta \Omega|\Omega)|^2 = \prod_{k\,1}^{\infty} [1 - 4x_k(1-x_k)\sin^2(\lambda_k\theta/2)].$$

Now $\{W_\theta\}_{\theta\in\mathbb{R}}$ is strongly continuous in θ; therefore

$$\theta \to |(W_\theta\Omega|\Omega)|^2 = \prod_{k\,1}^{\infty} [1 - 4x_k(1-x_k)\sin^2(\lambda_k\theta/2)].$$

is continuous $\forall \theta \in \mathbb{R}$. Let us call

$$f_k(\theta) = 1 - 4x_k(1-x_k)\sin^2(\lambda_k\theta/2), \quad P(\theta) = \prod_{k\,1}^{\infty} f_k(\theta).$$

We have $P(0) = 1$ and $\theta \to P(\theta)$ is continuous $\forall \theta \in \mathbb{R}$,

$$0 \le P(\theta) \le f_k(\theta) \le 1, \quad \forall k \in \mathbb{N},\ \forall \theta \in \mathbb{R},$$

and

$$\tfrac{2}{3}[1 - f_k(\theta)] \le |\mathrm{Log} f_k(\theta)|, \quad \forall k \in \mathbb{N},\ \forall \theta \in \mathbb{R},$$

(Log is Neper logarithm)

and

$$\mathrm{Log}\,P(\theta) = \sum_{k\,1}^{\infty} \mathrm{Log} f_k(\theta) \quad \text{for small } \theta\text{'s}.$$

Let us call

$$S(\theta) = \sum_{k\,1}^{\infty} [1 - f_k(\theta)] < +\infty \quad \text{for } P(\theta) \ne 0;$$

i.e., in a neighborhood of 0

$$\tfrac{2}{3} S(\theta) \le -\mathrm{Log}\,P(\theta) \quad \text{for } |\theta| \le \theta_0 < 1;$$

moreover,

$$\tfrac{2}{3} S_n(\theta) = \tfrac{2}{3}\sum_{k\,1}^{n} [1 - f_k(\theta)] \le \tfrac{2}{3} S(\theta) \le -\mathrm{Log}\,P(\theta).$$

Now, on $[-\theta_0, +\theta_0]$, $\theta \to -\mathrm{Log}\,P(\theta)$ is an integrable function, and S is measurable as a pointwise limit of measurable functions. Hence S is integrable on $[-\theta_0, +\theta_0]$. We take now $\theta \in [-\theta_0, +\theta_0]$,

$$S(\theta) = \sum_{k\,1}^{\infty} r_k\left(\frac{1-\cos\lambda_k\theta}{2}\right) < \infty,$$

$$F_n(\theta) = \int_0^\theta S_n(t)\,dt = \sum_{k\,1}^{n}\left(\frac{r_k\theta}{2} - \frac{\sin(\lambda_k\theta)}{2\lambda_k} r_k\right) \le \int_0^\theta S(t)\,dt.$$

Let

$$F(\theta) = \int_0^\theta S(t)\,dt,$$

$$\int_0^\theta F_n(t)\,dt = \sum_{k\,1}^{n}\left(\frac{r_k\theta^2}{4} + r_k\frac{\cos(\lambda_k\theta)-1}{2\lambda_k^2}\right) \le \int_0^\theta F(t)\,dt < +\infty.$$

(B) $\sum_{k\,1}^{\infty} r_k\left(\dfrac{1-\cos(\lambda_k\theta)}{2\lambda_k^2}\right) < +\infty$ since $|\lambda_k| \ge 1$.

(C) $\sum_{k\,1}^{\infty}\left(\dfrac{r_k\theta^2}{4} + r_k\dfrac{\cos(\lambda_k\theta)-1}{2\lambda_k^2}\right)$

absolutely converges, since

$$\sum_{k\,1}^{n}\left|\frac{r_k\theta^2}{4} + r_k\frac{\cos(\lambda_k\theta)-1}{2\lambda_k^2}\right| = \sum_{k\,1}^{n}\int_0^\theta |g_k(t)|\,dt \le \int_0^\theta F(t)\,dt < +\infty.$$

with $g_k(\theta) = \int_0^\theta [1 - f_k(t)]\,dt$

Obviously the sum of (B) and (C) shows that

$$\tfrac{1}{4}\theta^2 \sum_{k\,1}^{\infty} r_k < +\infty \quad \text{with } \theta \ne 0;$$

hence $\sum_{k\,1}^{\infty} r_k < +\infty$. ∎

Lemma 2.2: If $f : \mathbb{R} \to \mathbb{R}$, $f(0) = 0$, f differentiable at 0 and $f'(0) = 1$, $u_k \in \mathbb{R}$, $(u_k)_{k\in\mathbb{N}}$ bounded, $r_k \ge 0\ \forall k \in \mathbb{N}$, then

$$\left(\exists I \in \mathcal{V}_{\mathbb{R}}(0) \text{ and } \forall \theta \in I,\ \sum_{k\,1}^{\infty} r_k[f(u_k\theta)]^2 < +\infty\right) \Longleftrightarrow \sum_{k\,1}^{\infty} r_k u_k^2 < +\infty.$$

Proof: If $J \in \mathcal{V}_{\mathbb{R}}(0)$ is such that $x \in J \Rightarrow |[f(x)/x] - 1| < \frac{1}{2}$,

$$\tfrac{1}{2}x < f(x) < \tfrac{3}{2}x,$$

and $I \in \mathcal{V}_{\mathbb{R}}(0)$ is such that $\forall \theta \in I$, $\theta u_k \in J$, $\forall k \in \mathbb{N}$, then

$$\tfrac{1}{4}\theta^2 \sum_{k\,1}^{\infty} r_k u_k^2 \le \sum_{k\,1}^{\infty} r_k[f(u_k\theta)]^2 \le \tfrac{9}{4}\theta^2 \sum_{1}^{\infty} r_k u_k^2. \quad ∎$$

Now, we return to the proof of necessity. Let $\theta \in \mathbb{R}$ such that $\sum_{k\in\mathbb{N}} x_k(1-x_k)\sin^2(\lambda_k\theta/2) = +\infty$.

Let $u_k(\theta) = B(\cos(\lambda_k\theta/2)\psi_k^1 - \sin(\lambda_k\theta/2)\psi_k^2)B(\psi_k^1)$.

$$E_{n,m} = \bigoplus_{k=n}^{m} H_k, \quad u_{n,m}(\theta) = \prod_{k=n}^{m} u_k(\theta).$$

$$\omega_\Omega(x) = (\Omega|\pi_\Omega(x)\Omega), \quad \forall x \in \mathfrak{A} \tag{2.1}$$

We have

$$\forall x \in \mathfrak{A}(E_{n,m}, s), \quad \omega_\Omega(x) = \omega_\Omega \circ \tau_\theta(u_{n,m}(\theta)x u_{n,m}^*(\theta)).$$

Since

$$B(\psi)B(\varphi)B(\psi) = B(2s(\varphi,\psi)\psi - \varphi) = B(S_\psi\varphi),$$

S_ψ the symmetry with regard to ψ ($\|\psi\| = 1$).

For any $\varphi \in H_k$, $\tau_\theta(B(\varphi)) = B(e^{i\lambda_k\theta}\varphi) = B(R_{\lambda_k\theta}\varphi)$, $R_{\lambda_k\theta}$ the rotation of the argument $\lambda_k\theta$. Hence (2.1) holds.

Let us consider $\Theta_{n,m} = \prod_{k\,n}^{m}\Theta_k$; we shall note $\overline{\mathfrak{A}_e(H,s)}$ (resp. $\overline{\mathfrak{A}_0(H,s)}$) the C^*-algebra (resp. the closed vector-subspace) of $\overline{\mathfrak{A}(H,s)}$, generated by products of even (resp. odd) number of $B(\psi)$'s. Let us denote

$$\omega_{n,m} = \omega_\Omega|\mathfrak{A}_e(E_{n,m},s) \oplus \Theta_{1,n-1}\mathfrak{A}_0(E_{n,m},s),$$

$$\pi_{n,m} = \overset{\circ}{\bigotimes}{}_{n}^{m}{}_{k}\ \pi_k \quad \text{(tensor product "à la Powers"}^{3}\text{)},$$

$$\Omega_{n,m} = \bigotimes_n^m \Omega_k.$$

It is not difficult to see that

$\zeta : x + \Theta_{1,n-1} y \in \mathfrak{A}_e(E_{n,m}, s) \oplus \Theta_{1,n-1}\mathfrak{A}_0(E_{n,m}, s) \to x + y \in \mathfrak{A}(E_{n,m}, s)$ is a C^*-isomorphism and that $\omega_{n,m}(z) = (\Omega_{n,m}|\pi_{n,m}(\zeta(z))\Omega_{n,m}),\quad \forall z \in \mathfrak{A}_e(E_{n,m}, s) \oplus \Theta_{1,n-1}\mathfrak{A}_0(E_{n,m}, s)$. $\pi_{n,m}$ is an irreducible representation, and hence $\omega_{n,m}$ is a pure state. Lemma 2.4 of Ref. 7 implies $(\mathfrak{A}_e(E_{n,m}, s) \oplus \Theta_{1,n-1}\mathfrak{A}_0(E_{n,m}, s)$ is a C^*-algebra[8]):

$$\|(\omega_\Omega - \omega_\Omega \circ \tau_\theta)|\mathfrak{A}_e(E_{n,m}, s) \oplus \Theta_{1,n-1}\mathfrak{A}_0(E_{n,m}, s)\|$$
$$= 2[1 - |\omega_\Omega(u_{n,m}(\theta)|^2]^{1/2}$$
$$= 2\Big(1 - \prod_{k=n}^{m} [1 - 4x_k(1-x_k)\sin^2(\lambda_k\theta)/2)]\Big)^{1/2}.$$

Now

$$\sum_{k\in\mathbf{N}} x_k(1-x_k)\sin^2(\lambda_k\theta/2) = +\infty$$

implies

$$\prod_{i=n}^{\infty} [1 - 4x_k(1-x_k)\sin^2(\lambda_k\theta/2)] = 0,$$

i.e.,

$$\lim_{m,\infty} \prod_{i=n}^{m} [1 - 4x_k(1-x_k)\sin^2(\lambda_k\theta/2)] = 0.$$

Denote by $\mathfrak{A}(E_n, s)^c$ the commutant of $\mathfrak{A}(E_n, s)$ in $\mathfrak{A}$. Then

$$\Big(E_n = \bigoplus_{k=1}^{n} H_k\Big), \quad E_n^\perp = \bigoplus_{k=n+1}^{\infty} H_k,$$

$\mathfrak{A}(E_n, s)^c = \mathfrak{A}_e(E_n^\perp, s) \oplus \Theta_{1,n}\mathfrak{A}_0(E_n^\perp, s),$[9]

$$\mathfrak{A}_e(E_n^\perp, s) \oplus \Theta_{1,n}\mathfrak{A}_0(E_n^\perp, s) \supset \bigcup_{k\ n+1}^{\infty} [\mathfrak{A}_e(E_{n+1,k}, s) \oplus \Theta_{1,n-1}\mathfrak{A}_0(E_{n+1,k}, s)].$$

Thus

$$\|(\omega_\Omega - \omega_\Omega \circ \tau_\theta)|\mathfrak{A}(E_n, s)^c\|$$
$$= \|(\omega_\Omega - \omega_\Omega \circ \tau_\theta)|\mathfrak{A}_e(E_n^\perp, s) \oplus \Theta_{1,n-1}\mathfrak{A}_0(E_n^\perp, s)\|$$
$$\geq \lim_{k,\infty} \|(\omega_\Omega - \omega_\Omega \circ \tau_\theta)|\mathfrak{A}_e(E_{n+1,k}, s) \oplus \Theta_{1,n-1}\mathfrak{A}_0(E_{n+1,k}, s)\|$$
$$= 2.$$

Now $E_{n+1} \supset E_n$ and $\bigcup_{k\in\mathbf{N}} E_k = \bigoplus_{k\in\mathbf{N}} H_k$, $\overline{\bigcup_{k\in\mathbf{N}} E_k} = H$.

Hence, by Lemma 2.1 of Ref. 7, ω_Ω and $\omega_\Omega \circ \tau_\theta$ are not unitarily equivalent; therefore, no unitary $U_\theta \in \mathfrak{L}(\mathcal{H}^\Omega)$ can exist such that, $\forall x \in \mathfrak{A}$,

$$\pi_\Omega(\tau_\theta(x)) = U_\theta \tau_\Omega(x) U_\theta^*;$$

τ_θ is not implementable for the representation π_Ω. ■

IV. OTHER PROPOSITIONS AND REMARKS

(1) Fix $\theta \in \mathbf{R}$; there exists a unitary operator $U_\theta \in \mathfrak{L}(\mathcal{H}^\Omega)$ such that

$$\pi_\Omega(\tau_\theta(x)) = U_\theta \pi_\Omega(x) U_\theta^* \quad \forall x \in \mathfrak{A}$$

if and only if

$$\sum_{k\in\mathbf{N}} x_k(1-x_k)\sin^2(\lambda_k\theta/2) < +\infty. \tag{IV.1}$$

Proof: If (IV.1) is true, the existence of U_θ is checked (see the beginning of Sec. III).

If such a U_θ exists, $U_\theta = e^{i\rho}V_\theta$, V_θ is the operator constructed (Sec. IIA)

$$U_\theta \in \mathfrak{L}(\mathcal{H}^\Omega), \quad U_\theta = e^{i\rho} \bigotimes_{k\in\mathbf{N}} V_{k,\theta}.$$

$U_\theta\Omega = e^{i\rho}(V_{k,\theta}\Omega_k)_k \in \mathcal{H}^\Omega$, therefore $(V_{k,\theta}\Omega_k)_k \approx (\Omega_k)_k$ which implies $\sum_{k\in\mathbf{N}} |(V_{k,\theta}\Omega_k|\Omega_k) - 1| < +\infty$.

Recall that $\arg(V_{k,\theta}\Omega_k|\Omega_k) = 0$; hence

$$(V_{k,\theta}\Omega_k|\Omega_k) = |(V_{k,\theta}\Omega_k|\Omega_k)|$$

and

$$\sum_{k\in\mathbf{N}} x_k(1-x_k)\sin^2(\lambda_k\theta/2) < +\infty. \qquad ■$$

(2) Let

$$\mathfrak{N}_\Omega = \left\{\theta \in \mathbf{R} \;\middle|\; \begin{array}{l}\text{there exists a unitary operator}\\ U_\theta \in \mathfrak{L}(\mathcal{H}^\Omega) \text{ such that: } \forall x \in \mathfrak{A}\\ \pi_\Omega(\tau_\theta(x)) = U_\theta\pi_\Omega(x)U_\theta^*.\end{array}\right\}$$

$\mathfrak{N}_\Omega$ is an additive subgroup of $\mathbf{R}$.

Proof: Let $\theta_1, \theta_2 \in \mathfrak{N}_\Omega$. Then

$$\sum_{k\in\mathbf{N}} x_k(1-x_k)\sin^2(\lambda_k\theta_1/2) < +\infty$$

and

$$\sum_{k\in\mathbf{N}} x_k(1-x_k)\sin^2(\lambda_k\theta_2/2) < +\infty.$$

Let us set $r_k = x_k(1-x_k)$, $\varphi_k^1 = \lambda_k\theta_1/2$, $\varphi_k^2 = \lambda_k\theta_2/2$; $\sum_{k\in\mathbf{N}} r_k \sin^2(\varphi_k^1 + \varphi_k^2)$ converges, for

$$M = \sum_{k\in\mathbf{N}} r_k \sin^2(\varphi_k^1)\cos^2(\varphi_k^2) \leq \sum_{k\in\mathbf{N}} r_k \sin^2\varphi_k^1 < +\infty.$$

$$N = \sum_{k\in\mathbf{N}} r_k \sin^2(\varphi_k^2)\cos^2(\varphi_k^2) \leq \sum_{k\in\mathbf{N}} r_k \sin^2\varphi_k^2 < +\infty.$$

$$|L| \leq \sum_{k\in\mathbf{N}} 2r_k|\sin(\varphi_k^1)\sin(\varphi_k^2)\cos(\varphi_k^1)\cos(\varphi_1^2)| \leq \sum_{k\in\mathbf{N}} r_k[\sin^2(\varphi_k^1) + \sin^2(\varphi_k^2)] < +\infty.$$

Now

$$M + N + L = \sum_{k\in\mathbf{N}} r_k \sin^2(\varphi_k^1 + \varphi_k^2).$$

Obviously $\theta \in \mathfrak{N}_\Omega$ and $\theta \in \mathfrak{N}_\Omega \Longrightarrow -\theta \in \mathfrak{N}_\Omega$.

(3) If $\sum_{k\in\mathbf{N}} x_k(1-x_k) < +\infty$ we shall say that representation π_Ω is a *discrete* one. Sec. IIIB1 implies that all the monoparticular evolutions are implementable for every discrete representation.

Statement: If π_Ω is *not* a discrete representation (i.e., $\sum_{k\in\mathbf{N}} x_k(1-x_k) = +\infty$) and if $\{\lambda_k\}_{k\in\mathbf{N}}$ has neither 0 nor infinite as accumulation points, then $\mathfrak{N}_\Omega$

$= a\mathbb{Z}_+, a \in \mathbb{R}_+$ ($\mathbb{Z}$ the additive group of the relative integers).

Proof: Except for a finite number of k's we have

$$\lambda_k \in [a'', b''] \cup [a', b'] \quad \text{with } a'' < b'' < 0 < a' < b'.$$

We can omit a finite number of k's without changing $\mathfrak{N}_\Omega$ which is determined by the convergence of some series. The convergence of which is not changed by the suppression of a finite number of terms. Let us build a dividing decomposition of those intervals.

Let $a'_n = \frac{4}{3} a'_{n-1} = (\frac{4}{3})^n a'$, $I'_n = [a'_n = [a'_n, a'_{n+1}]$. A finite number of I'_n overlaps $[a', b']$.

Let $[r'_n, s'_n] = [\pi/3a'_n, \pi/2a'_{n+1}]$ which is a proper interval.

If $\mu \in I'_n$ and $\theta \in [r'_n, s'_n]$, then $\mu\theta \in [\pi/3, \pi/2]$. In the same way let us write

$$a''_n = \tfrac{3}{4} a''_{n-1} = (\tfrac{3}{4})^n a'' I''_n = [a''_n, a''_{n+1}].$$

A finite number of I''_n overlaps $[a'', b'']$.

Let $[r''_n, s''_n] = [\pi/2a''_n, \pi/3a''_{n+1}]$ which is a proper interval.

If $\mu \in I''_n$ and $\theta \in [r''_n, s''_n]$, then $\mu\theta \in [\pi/3, \pi/2]$. Let us denote $\{I_p\}_{1\le p\le m}$ and $\{[r_p, s_p]\}_{1\le p\le m}$ those intervals and let

$$L_p = \{k \in \mathbb{N} | \lambda_k \in I_p\}.$$

Then

$$\sum_{k\in\mathbb{N}} x_k(1-x_k) = \sum_{k\in \bigcup_p{}_1^m L_p} x_k(1-x_k) = +\infty.$$

If $k \in L_p$, then $\lambda_k \in I_p$, $\lambda_k\theta/2 \in [\pi/3, \pi/2]$ as soon as $\theta \in [2r_p, 2s_p]$; hence $\sin^2(\lambda_k\theta/2) \in [\frac{3}{4}, 1]$ and

$$\sum_{k\in\mathbb{N}} x_k(1-x_k)\sin^2(\lambda_k\theta/2) = +\infty.$$

Thus not any U_θ can exist for not any $\theta \in [r_p, s_p]$.

From that we conclude that $\mathfrak{N}_\Omega = a\mathbb{Z}$ for some $a \in \mathbb{R}_+$. ∎

(4.1) Definitions: As in Sec. IIIB2, we shall denote $\overline{\mathfrak{A}_0(H,s)}$ the closed vector subspace of $\mathfrak{A}$ generated by products of odd number of $B(\psi)$'s.

A state ω on $\mathfrak{A}$ will be called *quasifree*[3,10,1] when $\omega|\overline{\mathfrak{A}_0(H,s)} = 0$,

$$\omega\Big(\prod_{i\,1}^{2n} B(\varphi_i)\Big) = \sum_{\substack{i_1<i_2<\cdots<i_n\\ i_k<j_k}} \epsilon_\sigma \prod_{k\,1}^{n} \omega(B(\varphi_{i_k})B(\varphi_{j_k})),$$

ϵ_σ being the parity of the permutation σ

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & 2n-1 & 2n \\ i_1 & j_1 & \cdots & i_n & j_n \end{pmatrix}.$$

Let us call $\omega_\Omega(x) = (\Omega|\pi_\Omega(x)\Omega)$, $x \in \mathfrak{A}$, with

$$\Omega = \bigotimes_{k\in\mathbb{N}} \begin{pmatrix}\alpha_k\\ \beta_k\end{pmatrix}.$$

Accordingly with Sec. 4.3 below we call ω_Ω a *discrete state* iff $\sum_{k\in\mathbb{N}} x_k(1-x_k) < +\infty$ ($x_k = |\alpha_k|^2$).

(4.2) Lemma: ω_Ω is quasifree if and only if $\alpha_k\beta_k = 0$, $\forall k \in \mathbb{N}$.

Proof: Suppose ω_Ω is quasifree.

$$\left.\begin{aligned}\omega_\Omega(B(\psi_1^1)) &= 2\,\mathrm{Re}(\alpha_1\overline{\beta_1}) = 0\\ \omega_\Omega(B(\psi_1^2)) &= -2\,\mathrm{Im}(\alpha_1\overline{\beta_1}) = 0\end{aligned}\right\}$$

$\Longrightarrow \alpha_1\beta_1 = 0$ and $|\alpha_1|^2 - |\beta_1|^2 = \pm 1$;

hence

$$\left.\begin{aligned}\omega_\Omega(B(\psi_k^1)) &= \pm 2\,\mathrm{Re}(\alpha_k\overline{\beta_k}) = 0\\ \omega_\Omega(B(\psi_k^2)) &= \mp 2\,\mathrm{Im}(\alpha_k\overline{\beta_k}) = 0\end{aligned}\right\} \Longrightarrow \alpha_k\beta_k = 0.$$

Conversely, suppose $\forall k \in \mathbb{N}$, $\alpha_k\beta_k = 0$. Let $y = \prod_{k\,1}^{n} y_k$ with $y_k = B(\psi_k^1)B(\psi_k^2)$ or $y_k = B(\psi_k^1)$ or $y_k = B(\psi_k^2)$; if $y \in \overline{\mathfrak{A}_0(H,s)}$, at least there exists a $k_0 \in \mathbb{N}$ such that $y_{k_0} = B(\psi_{k_0}^j)$, $j = 1$ or $j = 2$, and

$$\omega_\Omega(y) = (\Omega|\pi_\Omega(y)\Omega) = \prod_{l\in\mathbb{N}} (\Omega_l|\pi_l(y_l)\Xi_l\Omega_l),$$
$$\Xi_l = I_l \text{ or } \sigma_l^3.$$

From

$$\left.\begin{aligned}(\Omega_{k_0}|\pi_{k_0}(B(\psi_{k_0}^j))\Omega_{k_0}) &= \pm 2{}^{\mathrm{Re}}_{\mathrm{Im}}(\alpha_{k_0}\overline{\beta_{k_0}}) = 0\\ (\Omega_{k_0}|\pi_{k_0}(B(\psi_{k_0}^j))\sigma_{k_0}^3\Omega_{k_0}) &= 2i{}^{\mathrm{Im}}_{\mathrm{Re}}(\alpha_{k_0}\overline{\beta_{k_0}}) = 0\end{aligned}\right\} \times \begin{pmatrix} j = 1 \text{ higher position}\\ j = 2 \text{ lower position}\end{pmatrix}$$

we deduce $\omega_\Omega|\overline{\mathfrak{A}_0(H,s)} = 0$.

Moreover,

$$\omega_\Omega\Big(\prod_{k\,1}^{n} B(\psi_k^1)B(\psi_k^1)\Big) = \prod_{k\,1}^{n} \omega_\Omega(B(\psi_k^1)B(\psi_k^2)). \quad ∎$$

(4.3) Proposition: There exists a quasifree state $\omega_{\Omega'}$ unitarily equivalent to ω_Ω iff ω_Ω is a discrete state.

Proof: Suppose ω_Ω is unitarily equivalent to a quasifree state $\omega_{\Omega'}$ with

$$\Omega' = \bigotimes_{k\in\mathbb{N}} \begin{pmatrix}\alpha'_k\\ \beta'_k\end{pmatrix}, \quad \alpha'_k\beta'_k = 0, \quad \forall k \in \mathbb{N}.$$

Recall that ω_Ω and $\omega_{\Omega'}$ are unitarily equivalent iff (Sec. II, Proposition)

$$\sum_{k\in\mathbb{N}} [1 - |(\Omega_k|\Omega'_k)|] < +\infty,$$

which is equivalent to $\exists\, M, L \subset \mathbb{N}|$, $M \cup L = \mathbb{N}$, $M \cap L = \emptyset$

$$\sum_{k\in M} (1 - |\alpha_k|) + \sum_{k\in L} (1 - |\beta_k|) < +\infty,$$

$$\sum_{k\in M} (1 - \sqrt{x_k}) + \sum_{k\in L} (1 - \sqrt{1-x_k}) < +\infty$$

which implies that: $\prod_{k\in M}\sqrt{x_k}$ converges and is different from 0, therefore so does $\prod_{k\in M} x_k$, $\sum_{k\in M} (1-x_k) < +\infty$; $\prod_{k\in L}\sqrt{1-x_k}$ converges and is different from 0, therefore so does $\Pi_{k\in L}(1-x_k)$, $\sum_{k\in L} x_k < +\infty$; so $\sum_{k\in \mathrm{N}} x_k(1-x_k) < +\infty$.

Conversely, if $\sum_{k\in\mathrm{N}} x_k(1-x_k) < +\infty$, let

$$M = \{k \in \mathrm{N} \mid x_k > \tfrac{1}{2}\} \qquad \sum_{k\in M} (1-x_k) < +\infty,$$

$$L = \mathrm{N} - M, \qquad \sum_{k\in L} x_k < +\infty,$$

which implies $\Pi_{k\in M} x_k$ converges and is different from 0 such as $\Pi_{k\in L}(1-x_k)$ and hence $\Pi_{k\in M}\sqrt{x_k}$ and $\Pi_{k\in L}\sqrt{1-x_k}$. In other words

$$\sum_{k\in M}(1-\sqrt{x_k}) + \sum_{k\in L}(1-\sqrt{1-x_k}) < +\infty.$$

Calling

$$\Omega' = \bigotimes_{k\in\mathrm{N}} \begin{pmatrix}\alpha'_k\\ \beta'_k\end{pmatrix} \quad \text{with} \begin{cases}\alpha'_k = 1,\ \beta'_k = 0 & \text{if } k \in M\\ \alpha'_k = 0,\ \beta'_k = 1 & \text{if } k \in L'\end{cases}$$

we have that $\sum_{k\in\mathrm{N}} [1 - |(\Omega_k|\Omega')|] < +\infty$ and the quasifree state $\omega_{\Omega'}$ is unitarily equivalent to ω_Ω. ■

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*Attaché de Recherche–C.N.R.S.

†Maître de Conférence–Université de Provence.

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Erratum: Neutron transport equations with spin-orbit coupling [J. Math. Phys. 14, 97 (1973)]

L. M. Tannenwald

Department of Physics, University of California, Berkeley, California 94720
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Eq. (2.26): F should be a subscript.

Eq. (3.15): The square brackets of the integral should contain

$$C_0\Pi + C_1\hat{u}'n - C_2(\hat{u}' \wedge \Pi) + C_3(\hat{u}' \wedge \hat{u}' \wedge \Pi).$$

Erratum: A stochastic Gaussian beam [J. Math. Phys. 14, 84 (1973)]

G. C. Papanicolaou, D. McLaughlin, and R. Burridge

Courant Institute of Mathematical Sciences, New York University, New York, New York 10012
(Received 6 March 1973)

The third line of Eq. (3.13) should be:

$$-\frac{1}{2}\int_0^\infty R(s)\sin(2s)ds\,\frac{\partial P^{(0)}}{\partial\chi} - 2\gamma\,\frac{\coth\theta}{\sinh\theta}\,\frac{\partial^2 P^{(0)}}{\partial\chi\partial\phi}.$$

Errata: Gauge transformations of second type and their implementations. I. Fermions [J. Math. Phys. 13, 2002 (1972)]

J. F. Gille and J. Manuceau

Centre de Physique Théorique, C.N.R.S., 13-Marseille 9ᵉ, France
(Received 6 March 1973)

The first equation of the second column, p. 2002 should read

$$\pi'(B(\psi^j_k)) = \bigotimes_{j\,1}^{k-1} \pi'_j(\epsilon_j\,\Theta_j) \otimes \pi'_k(B(\psi^j_k)) \bigotimes_{j\,k+1}^{\infty} Ij.$$

The second equation of the second column p. 2002 should read

$$\pi(B(\psi^j_k)) = \bigotimes_{l\,1}^{k-1} \pi_l(\Theta_l) \otimes \pi_k(B(\psi^j_k)) \otimes \bigotimes_{l\,k+1}^{\infty} I_l.$$

Inequality (ii) in 2. *Necessity* should read

$$\text{(ii)}\quad \sum_{k,|\lambda_k|\leq 1} x_k(1-x_k)\lambda^2{}_k < +\infty.$$

The inclusion of line 24 of the first column, p. 2005 should read

$$\mathfrak{A}_e(E_n^{\pm}, s) \oplus \Theta_{1,n}\mathfrak{A}_0(E_n^{\pm}, s) \supset \bigcup_{k}{}_{n+1}^{\infty} (\mathfrak{A}_e(E_{n+1,k}, s) \oplus \Theta_{1,n}\mathfrak{A}_0(E_{n+1,k}, s).$$

The inequality and equality of lines 27–30 of the first column, p. 2005 should read

$$\|(\omega_\Omega - \omega_\Omega \circ \tau_\theta)|\mathfrak{A}(E_n, s)^c\|$$
$$= \|(\omega_\Omega - \omega_\Omega \circ \tau_\theta)|\mathfrak{A}_e(E_n^{\pm}, s) \oplus \Theta_{1,n}\mathfrak{A}_0(E_n^{\pm}, s)\|$$
$$\geq \lim_{k,\infty} \|(\omega_\Omega - \omega_\Omega \circ \tau_\theta)|\mathfrak{A}_e(E_{n+1,k}, s)$$
$$\oplus \Theta_{1,n}\mathfrak{A}_0(E_{n+1,k}, s)\|.$$

Reference 10 should read E. Balslev and A. Verbeure, Commun. Math. Phys. 7, 55 (1968).